

# Tempered ultrafunctions

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## Abstract

Ultrafunctions are a particular class of functions defined on some non-Archimedean field. They provide generalized solutions to functional equations which do not have any solutions among the real functions or the distributions. In this paper we introduce a new class of ultrafunctions, called tempered ultrafunctions, which are somewhat related to the tempered distributions and present some interesting peculiarities.

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## 1 Introduction

In many circumstances, the notion of function is not sufficient to the needs of a theory and it is necessary to generalize it. Generalized functions are especially useful in making discontinuous functions more like smooth functions. They are applied extensively, especially in physics and engineering.

There is more than one recognized theory of *generalized functions*. We can recall, for example, the heuristic use of symbolic methods, called operational calculus. A basic book on operational calculus was Oliver Heaviside's *Electromagnetic Theory* of 1899 [16]. A very important steps in this topic was the introduction of the weak derivative and of the Dirac Delta function. The theory of Dirac and the theory of weak derivatives where unified by Schwartz in the beautiful theory of distributions (see e.g. [22] and [23]), also thanks to the previous work of Leray and Sobolev. Among people working in partial differential equations, the theory of Schwartz has been accepted as definitive (at least until now), but other notions of generalized functions have been introduced by Colombeau [12] and Mikio Sato [20], [21].

Having in mind the same purposes, in some recent papers, we have introduced and studied the notion of ultrafunction ([3], [5], [6], [7], [8], [9], [10], [11]). Ultrafunctions are a particular class of functions defined on a Non Archimedean field (we recall that a Non Archimedean field is an ordered field which contains infinite and infinitesimal numbers).

This paper continues such studies and introduces a new class of ultrafunctions called **tempered ultrafunctions**. This name comes from the fact that tempered ultrafunctions are somewhat related to the tempered distributions.

The theory of tempered ultrafunctions is based on four elements,  $V_\sigma, (^\circ), \oint, D$  :

- $V_\sigma$  is a  $\overline{\mathbb{R}}$ -linear subspace of  $\mathfrak{F}(\overline{\mathbb{R}}, \overline{\mathbb{C}})$  where  $\overline{\mathbb{R}}$  is a Non-Archimedean field,  $\overline{\mathbb{C}} = \overline{\mathbb{R}} \oplus i\overline{\mathbb{R}}$  and  $\mathfrak{F}(\overline{\mathbb{R}}, \overline{\mathbb{C}})$  is the set of all functions from  $\overline{\mathbb{R}}$  to  $\overline{\mathbb{C}}$ ;

•

$$(\circ) : \mathcal{S}'(\mathbb{R}) \rightarrow V_\sigma$$

is an injective linear map defined on the set of tempered distributions  $\mathcal{S}'(\mathbb{R})$ ;

•

$$\oint : V_\sigma \times V_\sigma \rightarrow \overline{\mathbb{C}}$$

is an Hermitian bilinear form;

•

$$D : V_\sigma \rightarrow V_\sigma$$

is a  $\overline{\mathbb{C}}$ -linear operator which extends the distributional derivative.

Some of the properties of the tempered ultrafunctions are described in the following theorem:

**Theorem 1** *The quadruplet  $\{V_\sigma, (\circ), \oint, D\}$  satisfies the following properties:*

1. if  $T \in \mathcal{S}'(\mathbb{R})$ , then,  $\forall \varphi \in \mathcal{S}(\mathbb{R})$

$$\oint T^\circ(x) \varphi^\circ(x) dx = \langle T, \varphi \rangle;$$

2. if  $f \in C^0(\mathbb{R})$  and  $\hat{f} \in L^1(\mathbb{R})$ , then

$$\forall x \in \mathbb{R}, \quad f^\circ(x) \sim f(x);$$

3. if  $\varphi \in \mathcal{S}(\mathbb{R})$  then

$$\forall x \in \mathbb{R}, \quad \varphi^\circ(x) = \varphi(x);$$

4. if  $T \in \mathcal{S}'(\mathbb{R})$ , and  $\partial$  is the distributional derivative, then

$$(\partial T)^\circ = DT^\circ$$

5. if  $f \in C^0(\mathbb{R})$  is a rapidly decreasing function, then

$$\oint f^\circ(x) dx = \int f(x) dx$$

6. if  $u, v \in V_\sigma$ ,

$$\oint Du(x)v(x) dx = - \oint u(x)Dv(x) dx;$$

7. let  $\mathcal{F} : \mathcal{S}'(\mathbb{R}) \rightarrow \mathcal{S}'(\mathbb{R})$  denote the distributional Fourier transform; then

$$\mathcal{F}[T](k) = \frac{1}{\sqrt{2\pi}} \oint T^\circ(x) (e^{ikx})^\circ dx$$

However one of the most interesting peculiarities of the ultrafunctions relies in the fact that they are functions and not functionals as the distributions. This fact implies that also  $\delta^2$ ,  $\sqrt{\delta}$  etc. are well defined functions in  $\mathfrak{F}(\overline{\mathbb{R}}, \overline{\mathbb{C}})$ .

## 1.1 Notations and definitions

We use this section to fix some notations and to recall some definitions:

- $\mathfrak{F}(X, Y)$  denotes the set of all functions from  $X$  to  $Y$ ;
- $\mathfrak{F}(E) = \mathfrak{F}(E, \mathbb{R})$ ;
- $C^k(\mathbb{R})$  denotes the set of functions in  $C(\mathbb{R})$  which have continuous derivatives up to the order  $k$ ;
- $\mathcal{D}(\mathbb{R})$  denotes the set of the infinitely differentiable functions with compact support;  $\mathcal{D}'(\mathbb{R})$  denotes the topological dual of  $\mathcal{D}(\mathbb{R})$ , namely the set of distributions on  $\mathbb{R}$ ;
- $\mathcal{S}(\mathbb{R})$  denotes the set of the infinitely differentiable functions rapidly decreasing with their derivatives;  $\mathcal{S}'(\mathbb{R})$  denotes the topological dual of  $\mathcal{S}(\mathbb{R})$ , namely the set of tempered distributions on  $\mathbb{R}$ ;
- $C_r^0(\mathbb{R})$  denotes the set of continuous functions slowly increasing (namely growing less than  $x^m$  for some  $m \in \mathbb{N}$ );
- if  $\mathbb{K}$  is a non-Archimedean field, then
  - $\text{mon}(x) = \{y \in \mathbb{K} : x \sim y\}$ ;
  - $\text{gal}(x) = \{y \in \mathbb{K} : x - y \text{ is a finite number}\}$ .

## 2 Some notions of Non-Archimedean mathematics

We believe that Non Archimedean Mathematics (NAM), namely, mathematics based on Non Archimedean Fields is very interesting, very rich and, in many circumstances, allows to construct models of the physical world in a more elegant and simple way. In the years around 1900, NAM was investigated by prominent mathematicians such as Du Bois-Reymond [13], Veronese [24], [25], David Hilbert [17], and Tullio Levi-Civita [14], but then it has been forgotten until the '60s when Abraham Robinson presented his Non Standard Analysis (NSA) [19]. We refer to Ehrlich [15] for a historical analysis of these facts and to Keisler [18] for a very clear exposition of NSA. Here, we will construch a model of NAM based on [2] and [4].

### 2.1 Non Archimedean Fields

In this section, we recall the basic definitions and facts regarding non-Archimedean fields. In the following,  $\mathbb{K}$  will denote an ordered field. We recall that such a field contains (a copy of) the rational numbers. Its elements will be called numbers.

**Definition 2** *Let  $\mathbb{K}$  be an ordered field. Let  $\xi \in \mathbb{K}$ . We say that:*

- $\xi$  is infinitesimal if, for all positive  $n \in \mathbb{N}$ ,  $|\xi| < \frac{1}{n}$ ;
- $\xi$  is finite if there exists  $n \in \mathbb{N}$  such as  $|\xi| < n$ ;
- $\xi$  is infinite if, for all  $n \in \mathbb{N}$ ,  $|\xi| > n$  (equivalently, if  $\xi$  is not finite).

**Definition 3** An ordered field  $\mathbb{K}$  is called *Non-Archimedean* if it contains an infinitesimal  $\xi \neq 0$ .

It's easily seen that all infinitesimal are finite, that the inverse of an infinite number is a nonzero infinitesimal number, and that the inverse of a nonzero infinitesimal number is infinite.

**Definition 4** A superreal field is an ordered field  $\mathbb{K}$  that properly extends  $\mathbb{R}$ .

It is easy to show, due to the completeness of  $\mathbb{R}$ , that there are nonzero infinitesimal numbers and infinite numbers in any superreal field. Infinitesimal numbers can be used to formalize a new notion of "closeness":

**Definition 5** We say that two numbers  $\xi, \zeta \in \mathbb{K}$  are *infinitely close* if  $\xi - \zeta$  is infinitesimal. In this case, we write  $\xi \sim \zeta$ .

Clearly, the relation " $\sim$ " of infinite closeness is an equivalence relation.

**Theorem 6** If  $\mathbb{K}$  is a superreal field, every finite number  $\xi \in \mathbb{K}$  is infinitely close to a unique real number  $r \sim \xi$ , called the **shadow** or the **standard part** of  $\xi$ .

Given a finite number  $\xi$ , we denote its shadow as  $sh(\xi)$ , and we put  $sh(\xi) = +\infty$  ( $sh(\xi) = -\infty$ ) if  $\xi \in \mathbb{K}$  is a positive (negative) infinite number.

**Definition 7** Let  $\mathbb{K}$  be a superreal field, and  $\xi \in \mathbb{K}$  a number. The *monad* of  $\xi$  is the set of all numbers that are infinitely close to it:

$$\mathbf{mon}(\xi) = \{\zeta \in \mathbb{K} : \xi \sim \zeta\},$$

and the *galaxy* of  $\xi$  is the set of all numbers that are finitely close to it:

$$\mathbf{gal}(\xi) = \{\zeta \in \mathbb{K} : \xi - \zeta \text{ is finite}\}$$

By definition, it follows that the set of infinitesimal numbers is  $\mathbf{mon}(0)$  and that the set of finite numbers is  $\mathbf{gal}(0)$ .

## 2.2 Sigma-convergence

When we take a limit of a sequence  $\varphi(n)$  for  $n \rightarrow \infty$ , the family of neighborhoods of  $\infty$  is the Frechet filter, namely the family of cofinite sets. In order to realize our program we need a finer topology based on an ultrafilter.

**Definition 8** *An ultrafilter  $\sigma$  on  $\mathbb{N}$  is a filter which satisfies the following property: if  $A \cup B = \mathbb{N}$ , then*

$$A \in \sigma \text{ or } B \in \sigma; \quad (1)$$

*We will refer to the sets in  $\sigma$  as **qualified sets**. If a property  $P(n)$  holds for every  $n$  in a qualified set we will say that  $P$  holds a.e. (almost everywhere).*

Now we define a topology on  $\mathbb{N} \cup \{\sigma\}$  as follows:  $N$  is a neighborhood of  $\sigma$  if and only if

$$N = Q \cup \{\sigma\} \quad \text{with} \quad Q \in \sigma \quad (2)$$

Thus  $\mathbb{N} \cup \{\sigma\}$  and  $\mathbb{N} \cup \{\infty\}$  are similar spaces but have different topologies: in fact  $\mathbb{N} \cup \{\sigma\}$  has a finer topology than  $\mathbb{N} \cup \{\infty\}$ . In any case you may think of  $\sigma$  as a "point at infinity".

**Definition 9** *Given a topological space  $X$ , we say that a sequence  $\varphi : \mathbb{N} \rightarrow X$  is  **$\sigma$ -convergent** to  $L \in X$  if and only if for any neighborhood  $N$  of  $L$ ,  $\exists Q \in \sigma$  such that,*

$$\forall n \in Q, \varphi_n \in N. \quad (3)$$

*We will write*

$$\lim_{n \uparrow \sigma} \varphi_n = L$$

Sometimes, the  $\sigma$ -convergence is called convergence along an ultrafilter. Notice that this topology satisfies this interesting property:

**Proposition 10** *If the sequence  $\varphi_n \in X$  has a converging subsequence, then it is  **$\sigma$ -converging**.*

**Proof:** Suppose that the net  $\varphi_n$  has a converging subnet to  $L \in \mathbb{R}$ . We a neighborhood  $N$  of  $L$  and we have to prove that  $Q_N \in \sigma$  where

$$Q_N = \{\lambda \in \mathbb{N} \mid \varphi_\lambda \in N\}.$$

We argue indirectly and we assume that

$$Q_N \notin \sigma$$

Then, by (1),  $N = \mathbb{N} \setminus (Q_N \cap X) \in \sigma$  and hence

$$\forall n \in N, \varphi_n \notin N.$$

This contradict the fact that  $\varphi_n$  has a subsequence which converges to  $L$ .

□

## 2.3 The fields of tempered hyperreal and hypercomplex numbers

In this section, we will construct a new Archimedean field, strictly related to  $\mathbb{R}^*$  which we will call field of the **tempered hyperreal numbers** and we will denote by  $\overline{\mathbb{R}}$ . We recall that  $\mathbb{R}^*$ , the field of hyperreal numbers, is the basic field in Nonstandard Analysis (see e.g. [19] or [18] for a very clear exposition; see [1], [2], or [4] for an exposition closer to the content of this paper). However, even if some ideas on Nonstandard Analysis are used, it is not necessary for the reader to be familiar with it in order to read this paper.

We define the family of **slowly increasing sequences** as follows

$$\mathfrak{F}_\tau(\mathbb{N}, \mathbb{R}) = \{\varphi \in \mathfrak{F}(\mathbb{N}, \mathbb{R}) \mid \exists k \in \mathbb{N}, |\varphi_n| \leq k + n^k\}$$

We say that a sequence  $\varphi_n$  is **rapidly increasing** if,  $\forall k \in \mathbb{N}$ ,  $\varphi_n \geq n^k$  for every  $n \geq \bar{n} \in \mathbb{N}$ .

Moreover the set of **rapidly decreasing sequences** (with respect to  $\sigma$ ) is defined as follows:

$$\mathfrak{I}(\mathbb{N}, \mathbb{R}) = \{\varphi \in \mathfrak{F}(\mathbb{N}, \mathbb{R}) \mid \forall k \in \mathbb{N}, \exists Q \in \sigma, \forall n \in Q, |\varphi_n| \leq n^{-k}\}$$

More in general, given an unbounded set  $E \subset \mathbb{R}$ , we define the family of **slowly increasing functions** as follows:

$$\mathfrak{F}_\tau(E, \mathbb{R}) = \{\varphi \in \mathfrak{F}(E, \mathbb{R}) \mid \exists k \in \mathbb{N}, \forall x \in E, |\varphi(x)| \leq k + |x|^k\}$$

and similarly we can define the rapidly increasing functions and rapidly decreasing functions.

We have the following result:

**Theorem 11** *There exists an ordered field  $\overline{\mathbb{R}} \supset \mathbb{R}$  ( $\overline{\mathbb{R}} \neq \mathbb{R}$ ) such that*

1. *Every sequence  $\varphi \in \mathfrak{F}_\tau(\mathbb{N}, \mathbb{R})$  has a unique limit*

$$\lim_{n \uparrow \sigma} \varphi_n = L \in \overline{\mathbb{R}}.$$

*which will be called  $\sigma$ -limit. Moreover every  $\xi \in \overline{\mathbb{R}}$  is the  $\sigma$ -limit of some sequence  $\varphi : \mathbb{N} \rightarrow \mathbb{R}$ .*

2. *If  $\varphi \in \mathfrak{I}(\mathbb{N}, \mathbb{R})$ , then*

$$\lim_{n \uparrow \sigma} \varphi_n = 0$$

3. *For all  $\varphi, \psi : \mathbb{N} \rightarrow \mathbb{R}$ :*

$$\begin{aligned} \lim_{n \uparrow \sigma} \varphi_n + \lim_{n \uparrow \sigma} \psi_n &= \lim_{n \uparrow \sigma} (\varphi_n + \psi_n); \\ \lim_{n \uparrow \sigma} \varphi_n \cdot \lim_{n \uparrow \sigma} \psi_n &= \lim_{n \uparrow \sigma} (\varphi_n \cdot \psi_n). \end{aligned}$$

**Proof.** Since  $\mathfrak{I}(\mathbb{N}, \mathbb{R})$  is an ideal in  $\mathfrak{F}_\tau(\mathbb{N}, \mathbb{R})$ , we have that

$$\overline{\mathbb{R}} = \frac{\mathfrak{F}_\tau(\mathbb{N}, \mathbb{R})}{\mathfrak{I}(\mathbb{N}, \mathbb{R})}$$

is a ring and it is not difficult to prove that it is an ordered field (namely  $\mathfrak{I}(\mathbb{N}, \mathbb{R})$  is a maximal ideal). If we set

$$\lim_{n \uparrow \sigma} \varphi_n = [\varphi]$$

we have that every sequence has a unique limit and if we identify a real number  $c \in \mathbb{R}$  with the equivalence class of the constant sequence  $[c]$ , then  $\mathbb{R} \subset \overline{\mathbb{R}}$ .

□

We have the following result:

**Theorem 12** *There is a topology on  $\overline{\mathbb{R}}$  consistent with the  $\sigma$ -limit. This topology will be called  $\sigma$ -topology.*

**Proof.** Probably, the simplest way to define this topology is giving the closure operator as follows: given a set  $E \subset \overline{\mathbb{R}}$ , we set,

$$\begin{aligned} D(E) &= \left\{ \xi \in \overline{\mathbb{R}} \setminus \mathbb{R} \mid \xi = \lim_{n \uparrow \sigma} \varphi_n \text{ where } \forall n \in \mathbb{N}, \varphi_n \in E \cap \mathbb{R} \right\} \text{ if } E \cap \mathbb{R} \neq \emptyset \\ D(E) &= \emptyset \text{ if } E \subset \overline{\mathbb{R}} \setminus \mathbb{R} = \emptyset \\ \overline{E} &= E \cup D(E) \end{aligned}$$

We will show that this operator satisfies the Kuratowski closure axioms:

1. Preservation of the empty set:  $\overline{\emptyset} = \emptyset$ .
2. Extensivity:  $E \subset \overline{E}$
3. Preservation of Binary Union:  $\overline{E \cup F} = \overline{E} \cup \overline{F}$
4. Idempotence:  $\overline{\overline{E}} = \overline{E}$

1 and 2 are trivial. Let us prove 3. By Def. 8, if  $\varphi_n \in (E \cup F) \cap \mathbb{R}$ , there exists a qualified set  $Q_\varphi$  such that  $\forall n \in Q_\varphi$ ,  $\varphi_n \in E \cap \mathbb{R}$  or  $\varphi_n \in F \cap \mathbb{R}$ ; then

$$\begin{aligned} D(E \cup F) &= \left\{ \lim_{n \uparrow \sigma} \varphi_n \mid \forall n \in \mathbb{N}, \varphi_n \in (E \cup F) \cap \mathbb{R} \right\} \\ &= \left\{ \lim_{n \uparrow \sigma} \varphi_n \mid \forall n \in Q_\varphi, \varphi_n \in E \cap \mathbb{R} \right\} \cup \left\{ \lim_{n \uparrow \sigma} \varphi_n \mid \forall n \in Q_\varphi, \varphi_n \in F \cap \mathbb{R} \right\} \\ &= D(E) \cup D(F) \end{aligned}$$



So, we have that

$$\begin{aligned}\overline{E \cup F} &= E \cup F \cup D(E \cup F) \\ &= E \cup F \cup D(E) \cup D(F) \\ &= \overline{E} \cup \overline{F}\end{aligned}$$

Now let us prove 4: Since  $D(E) \subset \overline{\mathbb{R}} \setminus \mathbb{R}$ , we have that  $D(D(E)) = \emptyset$ , and hence  $\overline{D(E)} = D(E)$ ; thus

$$\overline{\overline{E}} = \overline{E \cup D(E)} = \overline{E} \cup \overline{D(E)} = [E \cup D(E)] \cup D(E) = \overline{E}.$$

□

From now on, unless differently stated, the notation

$$\lim_{n \uparrow \sigma} \varphi_n$$

will denote the  $\sigma$ -limit of the sequence  $\varphi_n$  where the target space is  $\overline{\mathbb{R}}$  with its topology; the notation

$$\lim_{n \rightarrow \infty} \varphi_n$$

will denote the usual limit, where the target space is  $\mathbb{R}$  with the usual topology.

**Theorem 13** *Given a sequence  $\varphi_n \in \mathbb{R}$ , then we have one of the mutually exclusive possibility:*

1. *there is  $Q \in \sigma$  such that  $\varphi_n|_Q$  is slowly increasing; in this case we have that*

$$\lim_{n \uparrow \sigma} \varphi_n \in \overline{\mathbb{R}}$$

2. *there is  $Q \in \sigma$  such that  $\varphi_n|_Q$  is rapidly increasing; in this case we will write*

$$\lim_{n \uparrow \sigma} \varphi_n = +\overline{\infty}$$

3. *there is  $Q \in \sigma$  such that  $-\varphi_n|_Q$  is rapidly increasing; in this case we will write*

$$\lim_{n \uparrow \sigma} \varphi_n = -\overline{\infty}$$

**Proof:** It is an immediate consequence of the maximality of  $\sigma$ .

□

Using the above notation we have that  $\{\overline{\mathbb{R}}, +\overline{\infty}, -\overline{\infty}\}$  is the compactification of  $\overline{\mathbb{R}}$  analogous of the extended real line  $\{\mathbb{R}, +\infty, -\infty\}$ .

The following theorem whose proof is left to the reader shows some relation existing between the usual limit and the  $\sigma$ -limit.

**Theorem 14** If  $\varphi = \lim_{n \rightarrow \infty} \varphi_n$ , then

$$\lim_{n \uparrow \sigma} \varphi_n \sim \varphi.$$

Moreover, if  $\varphi - \varphi_n$  is a rapidly decreasing sequence, then

$$\lim_{n \uparrow \sigma} \varphi_n = \varphi.$$

Since in the next section, we will deal with complex valued function, the following trivial definition is needed.

**Definition 15** The complexification of  $\overline{\mathbb{R}}$  is called field of the tempered hyper-complex numbers and it is denoted by  $\overline{\mathbb{C}}$

### 3 The space of tempered ultrafunctions $V_\sigma$

#### 3.1 Definition of tempered ultrafunctions

We set

$$\beta_n := n\sqrt{\pi}; \quad \eta_n := \frac{\sqrt{\pi}}{n} \quad (4)$$

$$\Sigma_n = \{l\eta_n \mid l = -n^2, -n^2 + 1, -n^2 + 2, \dots, n^2 - 2, n^2 - 1\} \quad (5)$$

$$V_n = \text{Span} \{e^{ikx} \mid k \in \Sigma\}. \quad (6)$$

So,  $V_n$  satisfies the following properties:

- $V_n$  is a vector space of dimension  $2n^2$
- if  $f \in V_n$ ,  $f$  is periodic of period  $2\beta_n$

We now set

$$\Sigma = \left\{ \xi \in \overline{\mathbb{R}} \mid \exists \varphi \in \mathfrak{F}_\tau(\mathbb{N}, \mathbb{R}), \varphi_n \in \Sigma_n, \xi = \lim_{n \uparrow \sigma} \varphi_n \right\}$$

**Definition 16** Let  $E \subset \mathbb{R}$  and let  $\overline{E}$  denote its closure with respect to the  $\sigma$ -topology on  $\overline{\mathbb{R}}$ . A function

$$u : \overline{E} \rightarrow \overline{\mathbb{R}}$$

is called  $\sigma$ -limit function if there is a sequence of functions  $u_n \in \mathfrak{F}(E, \mathbb{R})$  such that  $\forall \xi \in \overline{E}$ ,

$$u(\xi) = \lim_{n \uparrow \sigma} u_n(x_n) \quad (7)$$

where

$$\lim_{n \uparrow \sigma} x_n = \xi; \quad x_n \in E.$$

The set of  $\sigma$ -limit functions will be denoted by  $\mathfrak{S}(\overline{E}, \overline{\mathbb{R}})$ . The set of tempered ultrafunction  $V_\sigma \subset \mathfrak{F}(\overline{\mathbb{R}}, \overline{\mathbb{R}})$  is the subset of the  $\sigma$ -limit functions such that  $\forall n \in \mathbb{N}$ ,

$$u_n \in V_n. \quad (8)$$

In the following, in order to denote the limit (7), we will use the following shorthand notation:

$$u = \lim_{n \uparrow \sigma} u_n.$$

**Remark 17** If  $f_n$  is a sequence of real functions, and  $\xi = \lim_{n \uparrow \sigma} x_n$ , then, by Th. 13,

$$f_\sigma(\xi) := \lim_{n \uparrow \sigma} f_n(x_n) \quad (9)$$

is always defined. However, if the sequence  $|f_n(x_n)|$  grows rapidly, we have that  $f_\sigma(\xi) = +\infty$  or  $f_\sigma(\xi) = -\infty$ .

Let us see an example of ultrafunction:

**The exponential ultrafunction.** If  $x \in \overline{\mathbb{R}}$ ,  $k \in \Sigma$ , we set

$$e^{ikx} = \lim_{n \uparrow \sigma} e^{ik_n x_n}$$

where  $\lim_{n \uparrow \sigma} x_n = x$  and  $\lim_{n \uparrow \sigma} k_n = k$ . If  $x, k \in \mathbb{R}$ , then  $e^{ikx}$  assume the same real values than the function  $e^{ikx}$ ; for this reason, they are denoted by the same symbols. In particular, for  $k = 0$ , we will denote by 1 the tempered ultrafunction identically equal to (the tempered hyperreal number) 1.

## 3.2 Basic operation with the tempered ultrafunctions

### 3.2.1 Derivative

The derivative of a tempered ultrafunction  $u = \lim_{n \uparrow \sigma} u_n$ ,  $u_n \in V_n$  is defined as follows:

$$Du = \lim_{n \uparrow \sigma} u'_n.$$

It is immediate to verify that the derivative satisfies the basic properties:

**Theorem 18** The derivative satisfies the following properties:

1. **Linearity:**  $D : V_\sigma \rightarrow V_\sigma$  is a linear operator over the field  $\overline{\mathbb{C}}$ ,

2. **Leibnitz Rule:** if  $u, v, D(uv), Duv, uDv \in V_\sigma$  then,

$$D(uv) = Duv + uDv$$

**Remark 19** Clearly, it is possible to define the derivative also of a  $\sigma$ -limit functions  $u = \lim_{n \uparrow \sigma} u_n$  provides that each  $u_n$  is differentiable.

### 3.2.2 Inner product

The inner product between two  $\sigma$ -limit functions is defined as follows:

$$(u|v) = \lim_{n \uparrow \sigma} \int_{-\beta_n}^{\beta_n} u_n(x) \overline{v_n(x)} dx$$

So  $V_\sigma$  is an Euclidean space over the field of the tempered hypercomplex numbers  $\overline{\mathbb{C}}$ . The inner product induces on the algebra of ultrafunctions the following "Euclidean" norm

$$\|u\| = \sqrt{(u|u)}$$

which takes values in  $\overline{\mathbb{R}}$ .

### 3.2.3 Integration

The integral of a  $\sigma$ -limit function

$$u = \lim_{n \uparrow \sigma} u_n, \quad u_n \in L_{loc}^1$$

is defined as follows:

$$\oint u(x) dx = \lim_{n \uparrow \sigma} \int_{-\beta_n}^{\beta_n} u_n(x) dx.$$

Notice that if  $n \mapsto \int_{-\beta_n}^{\beta_n} u_n(x) dx$  is a slowly increasing function, the above limit exists in  $\overline{\mathbb{R}}$  and we say that the integral converges; otherwise we say that the integral diverges to  $+\infty$  or to  $-\infty$ .

Using this definition, we have that

$$(u|v) = \oint u_n(x) \overline{v_n(x)} dx \tag{10}$$

where  $\bar{z}$  denotes the complex conjugate of  $z$ .

It is immediate to prove the following theorem:

**Theorem 20** *The integral satisfies the following properties:*

1.  $\oint : V_\sigma \rightarrow \overline{\mathbb{C}}$  is a linear functional over the field  $\overline{\mathbb{C}}$ ;
2. if  $u, v \in V_\sigma$ ,  $\oint Du(x)v(x)dx = - \oint u(x)Dv(x)dx$
3. if  $u \in V_\sigma$ ,  $\oint Du(x)dx = 0$

### 3.2.4 Hyperfinite sums

**Definition 21** A set  $\Gamma \subset \overline{\mathbb{R}}$  is called hyperfinite if there is a sequence of finite sets  $\Gamma_n \subset \mathbb{R}$  such that

$$\Gamma = \left\{ \xi \in \overline{\mathbb{R}} \mid \xi = \lim_{n \uparrow \sigma} \varphi_n, \text{ and } \forall n \in \mathbb{N}, \varphi_n \in \Gamma_n \right\}$$

**Definition 22** If  $\Gamma$  is a hyperfinite set and

$$u : \Gamma \rightarrow \overline{\mathbb{R}}; u = \lim_{n \uparrow \sigma} u_n; u_n : \Gamma_n \rightarrow \overline{\mathbb{R}},$$

is a  $\sigma$ -limit function, we set

$$\sum_{k \in \Gamma} u(k) = \lim_{n \uparrow \sigma} \sum_{k \in \Gamma_n} u_n(k)$$

$\sum_{k \in \Gamma} u(k)$  will be called the hyperfinite sum of the  $u(k)$ 's.

More in general, a family of  $\sigma$ -limit functions  $\{u_k\}_{k \in \Gamma}$  will be called hyperfinite if  $\Gamma \subset \overline{\mathbb{R}}$  is a hyperfinite set and the functional

$$k \mapsto u_k$$

is a  $\sigma$ -limit function. In this case, if  $x = \lim_{n \uparrow \sigma} x_n$ , we set

$$u(x) = \lim_{n \uparrow \sigma} \sum_{k \in \Gamma_n} u_k(x_n)$$

It is easy to check, just taking the appropriate limits from the finite case, that the hyperfinite sum commutes with the integral, namely

$$\sum_{j \in \Gamma} \left( \int u_j(x) dx \right) = \int \left( \sum_{j \in \Gamma} u_j(x) \right) dx.$$

## 4 Bases

### 4.1 The trigonometric basis

**Definition 23** An hyperfinite set of ultrafunctions  $\{e_j(x)\}_{j \in \Gamma}$  is called a basis for  $V_\sigma$  if every ultrafunction  $u \in V_\sigma$  can be written in a unique way by mean of an hyperfinite sum:

$$u(x) = \sum_{j \in \Gamma} u_j e_j(x)$$

A bases  $\{e_j(x)\}_{j \in \Gamma}$  is said orthonormal if

$$\oint e_j(x) e_k(x) dx = \delta_k^j.$$

By our construction we have that

$$\left\{ \frac{e^{ikx}}{\sqrt{2\beta_n}} \right\}_{k \in \Sigma_n}$$

is an orthonormal basis for  $V_n$ . Hence it is immediate to see that

$$\left\{ \frac{e^{ikx}}{\sqrt{2\beta}} \right\}_{k \in \Sigma} ; \quad \beta = \lim_{n \uparrow \sigma} \beta_n \quad (11)$$

is an orthonormal basis for  $V_\sigma$ . We will refer to it as the **trigonometric basis**. So we have the following result:

**Theorem 24** Any ultrafunction  $u \in V_\sigma$  can be represented as follows:

$$u(x) = \frac{1}{2\beta} \sum_{k \in \Sigma} \left( \oint u(t) e^{-ikt} dt \right) e^{ikx} = \frac{\eta}{2\pi} \sum_{k \in \Sigma} \left( \oint u(t) e^{-ikt} dt \right) e^{ikx} \quad (12)$$

Notice that the functions  $e^{ikx}$  are the eigenfunctions of the operator  $-iD_x$  with eigevalue  $k$ ; this operator restricted to  $V_\sigma$  is an Hermitian operator and  $\left\{ (2\beta)^{-1/2} e^{ikx} \right\}_{k \in \Sigma_n}$  is the relative orthonormal basis of eigenvalues.

Thus the above formula can be seen as the ultrafunctions variant of the relative spectral formula for selfadjoint operators. Also, as we will see later, it is strictly related to the Fourier transform.

## 4.2 The delta ultrafunction

**Definition 25** Given a number  $q \in \overline{\mathbb{R}}$ , we denote by  $\delta_q(x)$  an ultrafunction in  $V_\sigma$  such that

$$\forall v \in V_\sigma, \oint v(x)\delta_q(x)dx = v(q) \quad (13)$$

$\delta_q(x)$  is called Delta (or the Dirac) ultrafunction concentrated in  $q$ .

Let us see the first properties of the Delta ultrafunctions:

**Theorem 26** We have the following properties:

1. for every  $q \in \overline{\mathbb{R}}$  there exists a unique Delta ultrafunction concentrated in  $q$ ;
2.  $\overline{\delta_q(x)} = \delta_q(x)$
3. for every  $a, b \in \overline{\mathbb{R}}$ ,  $\delta_a(b) = \delta_b(a)$ ;
4.  $\|\delta_q\|^2 = \delta_q(q)$ .

**Proof.** (1) Let  $\{e_j\}_{j \in \Gamma}$  be an orthonormal basis of  $V_\sigma$ . We set

$$\delta_q(x) = \sum_{j \in \Gamma} e_j(q) \overline{e_j(x)} \quad \text{for any } q \in \mathbb{R} \quad (14)$$

Let us prove that  $\delta_q(x)$  actually satisfies (13). Let  $v(x) = \sum_{j \in \Gamma} v_j e_j(x)$  be any ultrafunction and  $q \in \mathbb{R}$ . Then

$$\begin{aligned} \oint v(x)\delta_q(x)dx &= \int \left( \sum_{j \in \Gamma} v_j e_j(x) \right) \left( \sum_{k \in \Gamma} e_k(q) \overline{e_k(x)} \right) dx = \\ &= \sum_{j,k \in \Gamma} v_j e_k(q) \oint e_j(x) \overline{e_k(x)} dx = \\ &= \sum_{j,k \in \Gamma} v_j e_k(q) \delta_j^q = \sum_{k \in \Gamma} v_k e_k(q) = v(q). \end{aligned}$$

So  $\delta_q(x)$ , defined by (14), is a Delta ultrafunction centered in  $q$ .

It is unique: if  $h_q(x)$  is another Delta ultrafunction centered in  $q$  then for every  $x \in \overline{\mathbb{R}}$ , we have:

$$\begin{aligned} \delta_q(x) - h_q(x) &= \oint (\delta_q(y) - h_q(y)) \delta_x(y) dy = \oint \delta_x(y) \delta_q(y) dy - \oint \delta_x(y) h_q(y) dy \\ &= \delta_x(q) - \delta_x(q) = 0 \end{aligned}$$

and hence  $\delta_q(x) = h_q(x)$  for every  $x \in \overline{\mathbb{R}}$ .

(2) We may assume that the  $e_j(q)$  are real functions in the sense that  $\overline{e_j(q)} = e_j(q)$ . Since, the  $e_j(q)$  are real functions, we have that  $\overline{\delta_q(x)} = \delta_q(x)$ .

$$(3) \quad \oint \delta_a(x) \delta_b(x) dx = \delta_b(a).$$

$$(4) \quad \|\delta_q\|^2 = \oint \delta_q(x) \delta_q(x) = \delta_q(q).$$

□

### 4.3 The canonical basis

**Theorem 27** For any  $a, b \in \Sigma$

$$\oint \delta_a(x) \delta_b(x) dx = \frac{\delta_b^a}{\eta}$$

where  $\delta_b^a$  denotes the  $\delta$  of Kronecker and

$$\eta = \lim_{n \uparrow \sigma} \eta_n$$

**Proof.** We have

$$\oint \delta_a(x) \delta_b(x) dx = \lim_{n \uparrow \sigma} \int_{-\beta_n}^{\beta_n} \delta_{n,a_n}(x) \delta_{n,b_n}(x) dx$$

where  $\delta_{n,a_n}$  and  $\delta_{n,b_n}$  are the approximations of  $\delta_a$  and  $\delta_b$  defined, using the orthonormal basis

$$\left\{ \frac{e^{ikx}}{\sqrt{2\beta_n}} \right\}_{k \in \Sigma_n},$$

as follows

$$\delta_{n,q}(x) = \frac{1}{2\beta_n} \sum_{k \in \Sigma_n} e^{ikq} e^{-ikx}. \quad (15)$$



Then we have

$$\begin{aligned}
\oint \delta_a(x) \delta_b(x) dx &= \oint \delta_a(x) \overline{\delta_b(x)} dx \\
&= \oint \left( \frac{1}{2\beta} \sum_{k \in \Sigma} e^{ika} e^{-ikx} \right) \left( \frac{1}{2\beta} \sum_{h \in \Sigma} e^{-ihb} e^{ihx} \right) dx \\
&= \frac{1}{(2\beta)^2} \left( \sum_{k, h \in \Sigma} e^{ika} e^{-ihb} \oint e^{ihx} e^{-ikx} dx \right) \\
&= \frac{1}{2\beta} \sum_{k \in \Sigma} e^{ika} e^{-ihb} \delta_h^k = \frac{1}{2\beta} \sum_{k \in \Sigma_n} e^{ika} e^{-ikb} \\
&= \frac{1}{2\beta} \sum_{k \in \Sigma} e^{ik(a-b)} = \lim_{n \uparrow \sigma} \frac{1}{2\beta_n} \sum_{k \in \Sigma_n} e^{ik(b_n - a_n)}
\end{aligned}$$

By our definitions we have that

$$\begin{aligned}
k &= l\eta_n, \quad l = -n^2, \dots, n^2 - 2, n^2 - 1 \\
b - a &= \lim_{n \uparrow \sigma} b_n - a_n \\
b_n - a_n &= L_n \eta_n, \quad L_n \in \mathbb{Z};
\end{aligned}$$

thus

$$\int_{-\beta_n}^{\beta_n} \delta_{n,a_n}(x) \delta_{n,b_n}(x) dx = \frac{1}{2\beta_n} \sum_{k \in \Sigma_n} e^{ikL_n \eta_n} = \frac{1}{2\beta_n} \sum_{l=-n^2}^{n^2-1} e^{ilL_n \eta_n^2}.$$

So, if  $b_n - a_n = L_n \eta_n = 0$ , then  $e^{\pi i L_n \eta_n^2} = 1$  and by (4)

$$\int \delta_{a_n}(x) \delta_{b_n}(x) dx = \frac{2n^2}{2\beta_n} = \frac{1}{\eta_n};$$

if  $b_n \neq a_n$ , since  $\eta_n^2 = \pi n^{-2}$  and  $L_n \in \mathbb{Z} \setminus \{0\}$ , we have that

$$\begin{aligned}
\int \delta_{a_n}(x) \delta_{b_n}(x) dx &= \frac{1}{2\beta_n} \sum_{l=-n^2}^{n^2-1} e^{ilL_n \eta_n^2} = \frac{1}{2\beta_n} \sum_{l=-n^2}^{n^2-1} e^{\pi i l L_n / n^2} \\
&= \frac{e^{i\pi L_n} - e^{-i\pi L_n}}{e^{\pi i L_n / n^2} - 1} \\
&= \frac{2i}{e^{\pi i L_n / n^2} - 1} \cdot \sin(\pi L_n) = 0
\end{aligned}$$

Concluding

$$\int \delta_{a_n}(x) \delta_{b_n}(x) dx = \frac{\delta_{a_n}^{b_n}}{\eta_n}$$

and hence

$$\oint \delta_a(x) \delta_b(x) dx = \frac{\delta_b^a}{\eta}.$$

□

By the above result and the definition of  $\delta_q(x)$ , it turns out that

$$\forall q, x \in \Sigma, \delta_q(x) = \oint \delta_q(t) \delta_x(t) dt = \frac{\delta_x^q}{\eta}.$$

Notice the different notation of the Dirac Delta ultrafunction  $\delta_q(x)$  and the Kronecker symbol  $\delta_x^q$ .

Moreover, we have that

$$\{\sqrt{\eta} \delta_q(x)\}_{q \in \Sigma}$$

is a orthonormal basis and we will refer to it as to the **canonical basis**. Then, using the canonical basis, an ultrafunction  $u$  can be represented as follows:

$$u(x) = \eta \sum_{q \in \Sigma} \left( \oint u(t) \delta_q(t) dt \right) \delta_q(x). \quad (16)$$

So, for every  $u \in V_\sigma$ , it makes sense to write

$$u(x) = \eta \sum_{q \in \Sigma} u(q) \delta_q(x). \quad (17)$$

and we get the following results:

**Corollary 28** *If  $u \in V_\sigma$  and  $\forall q \in \Sigma, u(q) = 0$ , then  $\forall x \in \overline{\mathbb{R}}, u(x) = 0$ .*

**Corollary 29** *Two tempered ultrafunctions which coincide on  $\Sigma$  are equal.*

**Corollary 30** *If  $u \in V_\sigma$ , then*

$$\oint u(x) dx = \eta \sum_{q \in \Sigma} u(q);$$

moreover, if  $u, v \in V_\sigma$ , then

$$\oint u(x) v(x) dx = \eta \sum_{q \in \Sigma} u(q) v(q);$$

Proof: The first inequality follows from (17) and the linearity of  $\oint$ ; also, by (17) and Th. 27, we have that

$$\begin{aligned} \oint u(x) v(x) dx &= \oint \left( \eta \sum_{q \in \Sigma} u(q) \delta_q(x) \right) \left( \eta \sum_{r \in \Sigma} v(r) \delta_r(x) \right) dx \\ &= \eta^2 \sum_{q, r \in \Sigma} u(q) v(r) \oint \delta_q(x) \delta_r(x) dx \\ &= \eta \sum_{q, r \in \Sigma} u(q) v(r) \delta_q^r = \eta \sum_{q \in \Sigma} u(q) v(q). \end{aligned}$$

□

So, in the world of tempered ultrafunctions, the integral is equal to the Riemann sums. However, this fact holds only for the ultrafunctions  $u \in V_\sigma$  and not for all the  $\sigma$ -limit functions.

## 5 The Fourier transform

### 5.1 Definition and main properties

**Definition 31** *The Fourier transform of an ultrafunction  $u \in V_\sigma$ ,  $\forall k \in \Sigma$ , is given by*

$$\mathfrak{F}[u](k) = \hat{u}(k) = \frac{1}{\sqrt{2\pi}} \oint u(x) e^{-ikx} dx.$$

By (17) and Corollary 29 it is possible to extend  $\hat{u}(k)$  to an ultrafunction defined on all  $\mathbb{R}$  setting  $\forall k \in \mathbb{R}$

$$\hat{u}(k) = \eta \sum_{h \in \Sigma} \hat{u}(h) \delta_q(k)$$

By Corollary 30,  $\hat{u}(k)$ ,  $k \in \Sigma$ , can be written as follows:

$$\hat{u}(k) = \frac{\eta}{\sqrt{2\pi}} \cdot \sum_{x \in \Sigma} u(x) e^{-ikx} \quad (18)$$

Moreover since  $\left\{ \frac{e^{ikx}}{\sqrt{2\beta}} \right\}_{k \in \Sigma} = \left\{ \sqrt{\frac{\eta}{2\pi}} e^{ikx} \right\}_{k \in \Sigma}$  is an orthonormal basis,  $\hat{u}(k)$  can be regarded, up to the constant  $\sqrt{\eta}$ , as the Fourier component of  $u(x)$  in this basis; namely we have that

$$\begin{aligned} u(x) &= \sum_{k \in \Sigma} \left( \oint u(y) \sqrt{\frac{\eta}{2\pi}} e^{-iky} dy \right) \sqrt{\frac{\eta}{2\pi}} e^{ikx} \\ &= \frac{\eta}{2\pi} \sum_{k \in \Sigma} \left( \oint u(y) e^{-iky} dy \right) e^{ikx} \\ &= \frac{\eta^2}{2\pi} \cdot \sum_{k \in \Sigma} \left( \sum_{y \in \Sigma} u(y) e^{-iky} \right) e^{ikx} \\ &= \frac{\eta}{\sqrt{2\pi}} \sum_{k \in \Sigma} \hat{u}(k) e^{ikx} \end{aligned}$$

**Examples:** By Th. 27, we have that

$$\oint e^{iqx} e^{-ixk} dx = 2\pi \delta_q(k). \quad (19)$$

and hence

$$\mathfrak{F} \left[ e^{iq(\cdot)} \right] (k) = \sqrt{2\pi} \delta_q(k)$$

Moreover, it is immediate to check that

$$\mathfrak{F} [\delta_q] (k) = \frac{1}{\sqrt{2\pi}} e^{-ikq}.$$

Thus the Fourier transform is a change of basis which sends the trigonometric basis into the canonical basis:

$$\left\{ \frac{e^{ikx}}{\sqrt{2\beta}} \right\}_{k \in \Sigma} \xrightarrow{\mathfrak{F}} \{ \sqrt{\eta} \delta_q(x) \}_{q \in \Sigma} \quad (20)$$

and hence it is a unitary operator and we have that

$$\mathfrak{F} [\delta_q] (k) = \frac{1}{\sqrt{2\pi}} e^{-ikq}.$$

Hence we have the following result:

**Theorem 32** *The Fourier transform*

$$\mathfrak{F} : V_\sigma \rightarrow V_\sigma$$

*is invertible and we have that*

$$\mathfrak{F}^{-1} [v] (x) = \frac{1}{\sqrt{2\pi}} \oint v(k) e^{ixk} dk = \frac{\eta}{\sqrt{2\pi}} \sum_{k \in \Sigma} v(k) e^{ikx}.$$

*Moreover*

$$\oint u(x) \overline{v(x)} dx = \oint \hat{u}(k) \overline{\hat{v}(k)} dk$$

**Proof:** It is an immediate consequence of (20).

□

## 5.2 The position operator

Now let us consider the operator

$$\check{x} : V_\sigma \rightarrow V_\sigma$$

defined as follows

$$\check{x}u(x) = \eta \sum_{q \in \Sigma} qu(q) \delta_q(x)$$

Thus if  $x \in \Sigma$ ,  $\check{x}u(x) = xu(x)$ ; however if  $x \notin \Sigma$ , it might happen that  $\check{x}u(x) \neq xu(x)$ . This possibility is excluded by (17) if  $xu(x) \in V_\sigma$ , but this fact is false for some  $u \in V_\sigma$ . Borrowing this name from quantum mechanics, we will call  $\check{x}$  the *position operator*.

We can extend to the world of ultrafunctions the well known relations between the position operator and  $-iD_x$ :

**Theorem 33** *For every  $u, v \in V_\sigma$ ,*

1.  $\mathfrak{F}[D_x u](k) = ik\hat{u}(k)$ ;
2.  $\mathfrak{F}[\check{x}u](k) = iD_k\hat{u}(k)$ .
3. *the operators  $\check{x}$  and  $-iD_x$  are Hermitian operators (in the sense that they are  $\sigma$ -limit of Hermitian operators in finite dimensional spaces) and the canonical and the trigonometric bases are the corresponding bases of eigenvalues.*

**Proof:** 1 - If  $k \in \Sigma$ , we have

$$\begin{aligned}\mathfrak{F}[D_x u](k) &= \frac{1}{\sqrt{2\pi}} \oint D_x u(x) e^{-ikx} dx = -\frac{1}{\sqrt{2\pi}} \oint u(x) D_x e^{-ikx} dx \\ &= \frac{ik}{\sqrt{2\pi}} \oint u(x) e^{-ikx} dx = ik\hat{u}(k)\end{aligned}$$

Thus, for a generic  $k \in \overline{\mathbb{R}}$ , we have

$$\mathfrak{F}[D_x u](k) = i\eta \sum_{q \in \Sigma} q \hat{u}(q) \delta_q(k) = ik\hat{u}(k)$$

2 - By Th. 30 and (18), we have that

$$\begin{aligned}\mathfrak{F}[\check{x}u](k) &= \frac{1}{\sqrt{2\pi}} \oint \check{x}u(x) e^{-ikx} dx = \frac{\eta}{\sqrt{2\pi}} \sum_{q \in \Sigma} qu(q) e^{-ikq} \\ &= \frac{\eta}{\sqrt{2\pi}} iD_k \sum_{q \in \Sigma} u(q) e^{-ikq} = iD_k \hat{u}(k)\end{aligned}$$

3 - Trivial.

□

## 6 The operator $(^\circ)$

We want to identify ultrafunctions with suitable functions or distributions; namely, we want to define a liner operator

$$(\circ) : \mathcal{S}'(\mathbb{R}) \rightarrow V_\sigma$$

such that

$$\begin{aligned} \forall \varphi \in \mathcal{S}(\mathbb{R}), \forall x \in \mathbb{R}, \varphi^\circ(x) &= \varphi(x) \\ \forall T \in \mathcal{S}'(\mathbb{R}), \forall \varphi \in \mathcal{S}(\mathbb{R}), \oint T^\circ(x) \varphi^\circ(x) dx &= \langle T, \varphi \rangle \\ \forall T \in \mathcal{S}'(\mathbb{R}), DT^\circ &= (DT)^\circ \end{aligned} \quad (21)$$

We will realize this program by two steps: first we define  $(^\circ)$  for  $C_\tau^0(\mathbb{R})$ , secondly we extend it to  $\mathcal{S}'(\mathbb{R})$ , the family of tempered distributions.

### 6.1 The $(^\circ)$ -operator for continuous slowly increasing functions

For what we have seen until now, the expression  $\oint f(x) \delta_q(x) dx$  makes sense when  $f$  is an ultrafunction; now we will define it for any  $f \in C_\tau^0$ . As it is easy to imagine, we set:

$$\oint f(x) \delta_q(x) dx := \lim_{n \uparrow \sigma} \int_{-\beta_n}^{\beta_n} f(x) \delta_{q,n}(x) dx \quad (22)$$

where  $\delta_{q,n}(x)$  is defined by (15),

Since  $f(y)$  is a slowly increasing function, the above sequence is slowly increasing and hence the integral converges.

**Definition 34** *If  $f \in C_\tau^0$ , we set*

$$f^\circ(x) = \eta \sum_{q \in \Sigma} \left( \oint f(y) \delta_q(y) dy \right) \delta_q(x);$$

where  $x = \lim_{n \uparrow \sigma} x_n \in \overline{\mathbb{R}}$ ; in particular, if  $x \in \Sigma$ ,

$$f^\circ(x) = \oint f(y) \delta_x(y) dy \quad (23)$$

The operator  $(^\circ)$  can also be characterized in the following way:

**Theorem 35** *The operator:*

$$(\circ) : C_\tau^0 \rightarrow V_\sigma$$

can be written as follows:

$$f^\circ(x) = \frac{1}{2\beta} \sum_{k \in \Sigma} \left( \oint f(y) e^{-iky} dy \right) e^{ikx} \quad (24)$$

$$= \lim_{n \uparrow \sigma} \frac{1}{2\beta_n} \sum_{k \in \Sigma_n} \left( \int_{-\beta_n}^{\beta_n} f(y) e^{-iky} dy \right) e^{ikx_n} \quad (25)$$

$$= \frac{1}{2\pi} \oint \left( \oint f(y) e^{-iky} dy \right) e^{ikx} dk \quad (26)$$

$$= \lim_{n \uparrow \sigma} \frac{1}{2\pi} \int_{-\beta_n}^{\beta_n} \left( \int_{-\beta_n}^{\beta_n} f(y) e^{-iky} dy \right) e^{ikx_n} dk \quad (27)$$

where

$$x = \lim_{n \uparrow \sigma} x_n$$

**Proof:** By Def. 34, we have that

$$\begin{aligned} f^\circ(x) &= \eta \sum_{q \in \Sigma} \left( \oint f(y) \delta_q(y) dy \right) \delta_q(x) \\ &= \lim_{n \uparrow \sigma} \eta_n \sum_{q \in \Sigma_n} \left( \int_{-\beta_n}^{\beta_n} f(y) \delta_{q,n}(y) dy \right) \delta_{q,n}(x) \\ &= \lim_{n \uparrow \sigma} P_n(f) \end{aligned}$$

where

$$P_n(f) := \eta_n \sum_{q \in \Sigma_n} \left( \int_{-\beta_n}^{\beta_n} f(y) \delta_{q,n}(y) dy \right) \delta_{q,n}(x)$$

is the orthogonal projection of  $f|_{[\beta_n, -\beta_n]}$  (extended by periodicity) on  $V_n$ . Then, representing this projection in the trigonometric basis we have that

$$P_n(f) := \frac{1}{2\beta_n} \sum_{k \in \Sigma_n} \left( \int_{-\beta_n}^{\beta_n} f(y) e^{-iky} dy \right) e^{ikx}$$

Then,

$$\begin{aligned} f^\circ(x) &= \lim_{n \uparrow \sigma} P_n(f) \\ &= \lim_{n \uparrow \sigma} \frac{1}{2\beta_n} \sum_{k \in \Sigma_n} \left( \int_{-\beta_n}^{\beta_n} f(y) e^{-iky} dy \right) e^{ikx} \\ &= \frac{1}{2\beta} \sum_{k \in \Sigma} \left( \oint f(y) e^{-iky} dy \right) e^{ikx} \end{aligned}$$

Thus (24) and (25) hold. In order to prove (26) we recall Corollary 30 and we get

$$\begin{aligned}
f^\circ(x) &= \frac{1}{2\beta} \sum_{k \in \Sigma} \left( \oint f(y) e^{-iky} dy \right) e^{ikx} \\
&= \frac{1}{2\beta\eta} \oint \left( \oint f(y) e^{-iky} dy \right) e^{ikx} dk \\
&= \frac{1}{2\pi} \oint \left( \oint f(y) e^{-iky} dy \right) e^{ikx} dk \\
&= \lim_{n \uparrow \sigma} \frac{1}{2\pi} \int_{-\beta_n}^{\beta_n} \left( \int_{-\beta_n}^{\beta_n} f(y) e^{-iky} dy \right) e^{ikx_n} dk
\end{aligned}$$

□

**Theorem 36** *If  $f \in C^0(\mathbb{R})$  is a rapidly decreasing function, then*

$$\oint f^\circ(x) dx = \int f(x) dx$$

**Proof:** Since

$$n \mapsto \int f(x) dx - \int_{-\beta_n}^{\beta_n} f(x) dx$$

is a rapidly decreasing sequence, from Th. 14, we have that

$$\int f(x) dx = \lim_{n \rightarrow \infty} \int_{-\beta_n}^{\beta_n} f(x) dx = \lim_{n \uparrow \sigma} \int_{-\beta_n}^{\beta_n} f(x) dx = \oint f^\circ(x) dx.$$

□

Next, we are going to check that (21) holds.

**Theorem 37** *If  $\varphi$  is a function such that  $\hat{\varphi} \in L^1(\mathbb{R})$ , then,  $\forall x \in \mathbb{R}$*

$$\varphi^\circ(x) \sim \varphi(x);$$

*moreover, if  $\varphi \in \mathcal{S}(\mathbb{R})$ , then,  $\forall x \in \mathbb{R}$*

$$\varphi^\circ(x) = \varphi(x).$$

**Proof:** If  $\hat{\varphi} \in L^1(\mathbb{R})$ , then,

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \lim_{n \rightarrow \infty} \int_{-\beta_n}^{\beta_n} \hat{\varphi}(k) e^{ikx_n} dk$$

and hence, from Th. 14, we have that

$$\varphi(x) \sim \frac{1}{\sqrt{2\pi}} \lim_{n \uparrow \sigma} \int_{-\beta_n}^{\beta_n} \hat{\varphi}(k) e^{ikx_n} dk = \varphi^\circ(x)$$



If  $\varphi \in \mathcal{S}(\mathbb{R})$ , we have that

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \int \hat{\varphi}(k) e^{ikx} dk = \frac{1}{\sqrt{2\pi}} \int_{-\beta_n}^{\beta_n} \hat{\varphi}(k) e^{ikx_n} dk + a_n$$

where  $a_n$  is rapidly decreasing. On the other hand,

$$\hat{\varphi}(k) = \frac{1}{\sqrt{2\pi}} \int \varphi(y) e^{-iky} dy = \frac{1}{\sqrt{2\pi}} \int_{-\beta_n}^{\beta_n} \varphi(y) e^{-iky} dy + b_n(k)$$

where  $\|b_n\|_{L^1}$  is rapidly decreasing. Then

$$\begin{aligned} \varphi(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\beta_n}^{\beta_n} \hat{\varphi}(k) e^{ikx_n} dk + a_n \\ &= \frac{1}{2\pi} \int_{-\beta_n}^{\beta_n} \left( \int_{-\beta_n}^{\beta_n} \varphi(y) e^{-iky} dy \right) e^{ikx_n} dk + \frac{1}{\sqrt{2\pi}} \int_{-\beta_n}^{\beta_n} b_n(k) e^{ikx_n} dk + \frac{1}{\sqrt{2\pi}} a_n \\ &= \frac{1}{2\pi} \int_{-\beta_n}^{\beta_n} \left( \int_{-\beta_n}^{\beta_n} \varphi(y) e^{-iky} dy \right) e^{ikx_n} dk + c_n \end{aligned}$$

where  $c_n$  is rapidly decreasing. Then, by Th. 14 and Th. 35, we have that

$$\begin{aligned} \varphi(x) &= \lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_{-\beta_n}^{\beta_n} \left( \int_{-\beta_n}^{\beta_n} \varphi(y) e^{-iky} dy \right) e^{ikx_n} dk \\ &= \lim_{n \uparrow \sigma} \frac{1}{2\pi} \int_{-\beta_n}^{\beta_n} \left( \int_{-\beta_n}^{\beta_n} \varphi(y) e^{-iky} dy \right) e^{ikx_n} dk = \varphi^\circ(x) \end{aligned}$$

□

## 6.2 The $(^\circ)$ -operator for tempered distributions

In order to extend  $(^\circ)$  to  $\mathcal{S}'(\mathbb{R})$  we recall the following result of Schwartz. If  $T \in \mathcal{S}'(\mathbb{R})$ , then there exists  $f \in C_\tau^0(\mathbb{R})$  and  $m \in \mathbb{N}$  such that

$$T = D^m f$$

This fact suggests immediately the following definition:

**Definition 38** Given  $T = D^m f \in \mathcal{S}'(\mathbb{R})$ , we set

$$T^\circ = D^m f^\circ$$

By the above definition, it follows immediately the following result:

**Theorem 39** *We have that*

$$\forall T \in \mathcal{S}'(\mathbb{R}), \forall \varphi \in \mathcal{S}(\mathbb{R}), \oint T^\circ(x) \varphi^\circ(x) dx = \langle T, \varphi \rangle \quad (28)$$

and

$$\forall T \in \mathcal{S}'(\mathbb{R}), DT^\circ = (DT)^\circ \quad (29)$$

**Proof:** First, let us prove (28):  $\forall \varphi \in \mathcal{S}(\mathbb{R}),$

$$\begin{aligned} \oint T^\circ(x) \varphi^\circ(x) dx &= \oint D^m f^\circ(x) \varphi^\circ(x) dx = (-1)^m \oint f^\circ(x) D^m \varphi^\circ(x) dx \\ &= (-1)^m \lim_{n \uparrow \sigma} \int_{-\beta_n}^{\beta_n} f(x) D^m \varphi(x) dx. \end{aligned}$$

Since  $f(x) D^m \varphi(x)$  is a rapidly decreasing function,

$$n \mapsto \int f(x) D^m \varphi(x) dx - \int_{-\beta_n}^{\beta_n} f(x) D^m \varphi(x) dx$$

is a rapidly decreasing sequence and hence, by Th. 14

$$\lim_{n \uparrow \sigma} \int_{-\beta_n}^{\beta_n} f(x) D^m \varphi(x) dx = \lim_{n \rightarrow \infty} \int_{-\beta_n}^{\beta_n} f(x) D^m \varphi(x) dx$$

and so

$$\begin{aligned} \oint T^\circ(x) \varphi^\circ(x) dx &= (-1)^m \lim_{n \rightarrow \infty} \int_{-\beta_n}^{\beta_n} f(x) D^m \varphi(x) dx \\ &= (-1)^m \int f(x) D^m \varphi(x) dx = \langle T, \varphi \rangle. \end{aligned}$$

Now let us prove (29). If  $T = D^m f$ , we have that

$$\begin{aligned} (DT)^\circ &= (DD^m f)^\circ = (D^{m+1} f)^\circ = D^{m+1} f^\circ \\ &= D(D^m f^\circ) = D(D^m f)^\circ = DT^\circ \end{aligned}$$

□

Some of the previous results can be summarized by theorem 1.

**Proof of Th. 1:** 1 follows from (28); 2 and 3 follow from Th. 37; 4 follows from (29); 5 follows from Th 36, 6 follows from Th 20, 7 follows from Def. 31 and (28).

□

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